On "Axiomatising Finite Concurrent Processes"

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Abstract

In his pioneering paper [14], Hennessy gave complete axiomatizations of Milner's observational congruence and of t-observational congruence which made use of an auxiliary operation to axiomatize parallel composition. Unfortunately, those axiomatizations turn out to be flawed due to the subtle interplay between Hennessy's auxiliary parallel operator and synchronization. The aim of this paper is to present correct versions of the equational characterizations given in [14]. Some of the problems which arise in giving operational semantics to the auxiliary operators used in [4, 6, 14] in the theory of congruences like Milner's observational congruence are also discussed.

Key words: Concurrent processes, observational congruence, t-observational congruence, equational logic.

1 Introduction

In his seminal paper [14], Matthew Hennessy has given complete axiomatizations of two behavioural congruences, namely those associated with Milner's *weak bisimulation equivalence* [19] and *t-observational equivalence* [14] (also known as *split-2 equivalence* [10] and *timed equivalence* [1]), over a simple language for concurrent processes. Paper [14] evolved from an early preprint, entitled "On the Relationship between Time and Interleaving", which dated back to 1981 and, in my opinion at least, did not receive the attention it deserved at the time of its first circulation.

Hennessy's "On the Relationship between Time and Interleaving" and its published version [14] have historically played an important role in the development of the theory of process algebras for at least two reasons. First, the equational characterization of observational congruence presented in these papers has been, to the best of my knowledge, the first one to use auxiliary operators in the axiomatization of CCS parallel composition [19]. At more or less the same time, J.A. Bergstra and J.W. Klop were working on a finite axiomatization of strong bisimulation e0Tdf999.7(0Td3Tdf999.xillgzatio equivalences, such as observational congruence and t-observational congruence. The aim of this note is to present correct versions of the axiomatizations given in [14]. In passing, I shall also comment on some of the issues involved in giving suitable operational semantics for the auxiliary operations of ACP in the setting of observational congruence and related congruences. I hope that this will make this paper a useful reference for researchers interested in complete axiomatizations of behavioural congruences.

2 An axiomatization of Hennessy's t-observational congruence

I assume that the reader is familiar with [14] and the basic notions on process algebras and bisimulation equivalence. The uninitiated reader is referred to the textbooks [19, 3] for extensive motivations and background. As this is not an introductory paper, I shall feel free to refer the reader to the motivations, definitions and results given in [14].

$$\frac{}{a.p \stackrel{S(a)}{\Rightarrow} a_S.p} \qquad \frac{}{a_S.p \stackrel{F(a)}{\Rightarrow} p} \qquad \frac{}{\mu.p \stackrel{\mu}{\Rightarrow} p}$$

$$\frac{s_1 \stackrel{e}{\Rightarrow} s_1'}{s_1 + s_2 \stackrel{e}{\Rightarrow} s_1'} \qquad \frac{s_1 \stackrel{e}{\Rightarrow} s_1'}{s_2 + s_1 \stackrel{e}{\Rightarrow} s_1'}$$

$$\frac{s_1 \stackrel{e}{\Rightarrow} s_1'}{s_1 \| s_2 \stackrel{e}{\Rightarrow} s_1' \| s_2} \qquad \frac{s_1 \stackrel{e}{\Rightarrow} s_1'}{s_2 \| s_1 \stackrel{e}{\Rightarrow} s_2 \| s_1'} \qquad \frac{s_1 \stackrel{e}{\Rightarrow} s_1'}{s_1 \not| \; s_2 \stackrel{e}{\Rightarrow} s_1' \| s_2}$$

$$\frac{s_1 \stackrel{a}{\Rightarrow} s_1', s_2 \stackrel{\bar{a}}{\Rightarrow} s_2'}{s_1 \| s_2 \stackrel{\tau}{\Rightarrow} s_1' \| s_2'} \qquad \frac{s_1 \stackrel{a}{\Rightarrow} s_1', s_2 \stackrel{\bar{a}}{\Rightarrow} s_2'}{s_1 \not| \; s_2 \stackrel{\tau}{\Rightarrow} s_1' \| s_2'}$$

$$\frac{s_1 \stackrel{\tau}{\Rightarrow} s_1', s_1' \stackrel{e}{\Rightarrow} s_2}{s_1 \stackrel{e}{\Rightarrow} s_2} \qquad \frac{s_1 \stackrel{e}{\Rightarrow} s_1', s_1' \stackrel{\tau}{\Rightarrow} s_2}{s_1 \stackrel{e}{\Rightarrow} s_2}$$

Figure 1: Operational rules for $\stackrel{e}{\Rightarrow}$

Unfortunately, however, the axiomatization presented in Figure 2 is incorrect. This is due to the fact that axiom (B2), which plays a vital role in the reduction of terms to

The key to the soundness of the above equation is the fact that the left-merge operation does not allow for synchronization between its operands. For example, the reader can easily adapt the aforementioned example showing the unsoundness of axiom (B2) to prove that a version of the above equation in terms of the communication merge is not valid in $\mathsf{P}_{\mathsf{ext}}/\approx_T^C$, *i.e.* that there are processes $p, q, r \in \mathsf{P}_{\mathsf{ext}}$ such that

$$(p \mid q) \mid r \quad \not\approx_T^C \quad p \mid (q \| r)$$

Synchronization between processes is described by the communication merge operator. In fact, left-merge and communication merge together allow one to describe equationally the behaviour of parallel composition and of Hennessy's $/\!\!/$ operator. The relevant equations are:

$$(x \,/\!\!/\, y)$$

$$
\begin{array}{llcl}
\text{A1} & (x+y)+z & = & x+(y+z) \\
\text{A2} & x+y & = & y+x \\
\text{A3} & x+x & = & x \\
\text{A4} & x+\mathbf{0} & = & x \\
 & & & \\
\text{LM1} & (x+y) \lfloor\!\lfloor z & = & x \lfloor\!\lfloor z + y \lfloor\!\lfloor z \\
\text{LM2} & (x \lfloor\!\lfloor y) \lfloor\!\lfloor z & = & x \lfloor\!\lfloor (y \| z) \\
\text{LM3} & x \lfloor\!\lfloor \mathbf{0} & = & x \\
\text{LM4} & \mathbf{0} \lfloor\!\lfloor x & = & \mathbf{0} \\
 & & & \\
\text{I1} & x+\tau.x & = & \tau.x \\
\text{I2} & \mu.\tau.x & = & \mu.x \\
\text{ILM1} & x \lfloor\!\lfloor (y+\tau.z) & = & x \lfloor\!\lfloor (y+\tau.z) + x \lfloor\!\lfloor z \\
\text{ILM2} & \tau.x \lfloor\!\lfloor y & = & \tau.(x \| y) \\
\text{ILM3} & x \lfloor\!\lfloor \tau.y & = & x \lfloor\!\lfloor y \\
 & & & \\
\text{CM1} & (x+y) \mid z & = & x \mid z + y \mid z \\
\text{CM2} & x \mid y & = & y \mid x \\
\text{CM3} & x \mid \mathbf{0} & = & \mathbf{0} \\
\text{CM4} & (a.x \lfloor\!\lfloor y) \mid (b.w \lfloor\!\lfloor z) & = & \begin{cases} \tau.(x \| y \| w \| z) & \text{if } a = \bar{b} \\ \mathbf{0} & \text{otherwise} \end{cases} \\
\text{CM5} & \tau.x \mid y & = & x \mid y \\
 & & & \\
\text{PAR} & x \| y & = & x \lfloor\!\lfloor y + y \lfloor\!\lfloor x + x \mid y \\
\text{HM} & x \not\| y & = & x \lfloor\!\lfloor y + x \mid y
\end{array}
$$

Figure 3: Complete equations for $\approx_T^C$ over $\mathsf{P}_{\mathsf{ext}}$

2.1 An axiomatization of observational congruence

As mentioned in the introduction, Hennessy's axiomatization of Milner's observational congruence in [14] was the first one to use an auxiliary operator to give an equational characterization of parallel composition. For the sake of clarity and in order to support the discussion to follow, I shall now recapitulate the definitions of weak bisimulation equivalence and its associated congruence.

The relation of *weak bisimulation equivalence* $\approx$ is defined as the largest symmetric relation over P which satisfies

$p \approx q$ iff for every $\mu \in \mathsf{Act}$, $p \stackrel{\mu}{\Rightarrow} p'$ implies

- $\mu = \tau$ and $p' \approx_T q$, or
- $q \stackrel{\mu}{\Rightarrow} q'$ for some q' such that $p' \approx_T q'$.

As usual, $\approx$ is not a congruence over P. The largest congruence relation contained in $\approx$ will be denoted by $\approx^C$ and will be referred to as *observational congruence*.

The key to the axiomatization of observational congruence presented in Theorem 1.3.4 of [14] is a version of Milner's *interleaving law* in terms of Hennessy's $|\!/$. This is the following conditional equation schema:

(X2) $\quad y = \sum \{\lambda_j . y_j \mid j \in J\}$ (J a finite index

$$
\begin{array}{lrcl}
\text{A1} & (x+y)+z & = & x+(y+z) \\
\text{A2} & x+y & = & y+x \\
\text{A3} & x+x & = & x \\
\text{A4} & x+\mathbf{0} & = & x \\
 & & & \\
\text{LM1} & (x+y) \mathbin{\|\!\_} z & = & x \mathbin{\|\!\_} z + y \mathbin{\|\!\_} z \\
\text{NLM2} & \mu.x \mathbin{\|\!\_} y & = & \mu.(x \| y) \\
\text{LM4} & \mathbf{0} \mathbin{\|\!\_} & &
\end{array}
$$

$$\frac{}{a.p \stackrel{S(a)}{\rightarrow} a_S.p} \qquad \frac{}{a_S.p \stackrel{F(a)}{\rightarrow} p} \qquad \frac{}{\mu.p \stackrel{\mu}{\rightarrow} p}$$

$$\frac{s_1 \stackrel{e}{\rightarrow} s_1'}{s_1 + s_2 \stackrel{e}{\rightarrow} s_1'} \qquad \frac{s_1 \stackrel{e}{\rightarrow} s_1'}{s_2 + s_1 \stackrel{e}{\rightarrow} s_1'}$$

$$\frac{s_1 \stackrel{e}{\rightarrow} s_1'}{s_1 \| s_2 \stackrel{e}{\rightarrow} s_1' \| s_2} \qquad \frac{s_1 \stackrel{e}{\rightarrow} s_1'}{s_2 \| s_1 \stackrel{e}{\rightarrow} s_2 \| s_1'} \qquad \frac{s_1 \stackrel{e}{\rightarrow} s_1'}{s_1 \mathbin{\|\!\lfloor} s_2 \stackrel{e}{\rightarrow} s_1' \| s_2} \qquad \frac{s_1 \stackrel{e}{\rightarrow} s_1'}{s_1 \not| \, s_2 \stackrel{e}{\rightarrow} s_1' \| s_2}$$

$$\frac{s_1 \stackrel{a}{\rightarrow} s_1', s_2 \stackrel{\bar{a}}{\rightarrow} s_2'}{s_1 \| s_2 \stackrel{\tau}{\rightarrow} s_1' \| s_2'} \qquad \frac{s_1 \stackrel{a}{\rightarrow} s_1', s_2 \stackrel{\bar{a}}{\rightarrow} s_2'}{s_1 \mid s_2 \stackrel{\tau}{\rightarrow} s_1' \| s_2'} \qquad \frac{s_1 \stackrel{a}{\rightarrow} s_1', s_2 \stackrel{\bar{a}}{\rightarrow} s_2'}{s_1 \not| \, s_2 \stackrel{\tau}{\rightarrow} s_1' \| s_2'}$$

Figure 5: Operational rules for $\stackrel{e}{\rightarrow}$

so-called *weak transition relations* over the language $\mathsf{P}_{\mathsf{ext}}$ in one step, so to speak. This is in contrast with the developments in, *e.g.*, [19], where the operational semantics of CCS is defined first in terms of single step transition relations. These concrete transition relations are then used in the definition of the weak transition relations, which capture the intuition that τ-labelled transitions correspond to invisible events. For easy reference, the defining rules of the one step transition relations, $\stackrel{e}{\rightarrow}$, for the language $\mathsf{P}_{\mathsf{ext}}$ are collected in Figure 5. The associated transition relations which abstract from internal τ-transitions are then usually defined by:

$$s \stackrel{e}{\Rightarrow} s' \Leftrightarrow \exists s_1, s_2 : \; s \stackrel{\tau}{\rightarrow}^{\star} s_1 \stackrel{e}{\rightarrow} s_2 \stackrel{\tau}{\rightarrow}^{\star} s'$$

where $\stackrel{\tau}{\rightarrow}^{\star}$ denotes the reflexive and transitive closure of the relation $\stackrel{\tau}{\rightarrow}$.

The process of abstraction from τ-labelled transitions is instead built in the definition of the transition relations $\stackrel{e}{\Rightarrow}$ by means of the rules

$$\frac{s_1 \stackrel{\tau}{\Rightarrow} s_1', s_1' \stackrel{e}{\Rightarrow} s_2}{s_1 \stackrel{e}{\Rightarrow} s_2} \qquad \frac{s_1 \stackrel{e}{\Rightarrow} s_1', s_1' \stackrel{\tau}{\Rightarrow} s_2}{s_1 \stackrel{e}{\Rightarrow} s_2}$$

It is easy to see that, for processes in $\mathsf{P}_{\mathsf{ext}}$ not containing occurrences of the communication merge and of Hennessy's $\not|$ 1e

Behavioural Congruences	**Suitable Transition Relations**
Observational congruences satisfying (I1)	Define the semantics of the auxiliary operators by giving rules which give the weak transition relations in one step, as in Figure 1.
Branching bisimulation	

the kind of behavioural congruence one wants to impose on terms. A full discussion of this point would lead me too far from the main aim of this paper. Thus, I shall just end by giving a short "recipe book" for giving semantics to the auxiliary operators discussed in this paper in the setting of some of the best-known semantic theories for processes, with pointers to references where they are discussed in detail. These may be found in Figure 6. I hope that they will be a useful reference for researchers interested in complete axiomatizations of behavioural congruences.

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