Averaging and Eliciting Expert Opinion[∗]

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Abstract

The paper considers the problem of averaging expert opinion when opinions are expressed quantitatively by belief functions in the sense of Glenn Shafer. Practical experience shows that experts usually di er in their exact quantitative assessments and some method of averaging is necessary. A natural requirement of consistency demands that the operations of averaging and combination, in the sense of Dempster's rule, should commute. Experience also shows that symmetric belief functions are di cult to distinguish in practice. By forming a quotient of the monoid of belief functions modulo the ideal of symmetric belief functions, we are left with an Abelian group with a natural scalar multiplication making it a real vector space. The elements of this quotient space correspond to what we call "regular" belief functions. This solves the averaging problem with arbitrary weights. The existence of additive inverses for regular belief functions means that contrary evidence can be treated without assuming the existence of complements. Opinions expressed by conditional judgements can be incorporated by lifting suitable measures from a quotient space to a numerator. The appendix describes a computer program for implementing these ideas in practice.

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2.1 Distributive Lattices

A partially ordered $\frac{1}{15}$ a set $\frac{1}{15}$ a set $\frac{1}{15}$ a $\frac{1}{15}$ a $\frac{1}{15}$ a $\frac{1}{15}$ a $\frac{1}{15}$ $a \le a$ $a \leq b$ n $b \leq a$ po a b $a \leq b$ n $b \leq c$ p_p $a \leq c$ o alb, c in A. A $_{\rm res}$ is S of a partially ordered set A is said to be an upper set $\,$ $\rm A$

2.2 Probability Measures on Distributive Lattices

Proposition 1 Every probability measure on a distributive lattice D has a unique extension to a probability measure on the Boolean algebra freely generated by D.

 nn o_{f the sign} $\frac{0}{\sqrt{2}}$ is $\frac{0}{\sqrt{2}}$ for $\frac{0}{\sqrt{2}}$ freely generated by $\frac{1}{\sqrt{2}}$ freely generated by $\frac{1}{\sqrt{2}}$ freely generated by $\frac{1}{\sqrt{2}}$ freely generated by $\frac{1}{\sqrt{2}}$ freely generated by a distribution D is $\mathsf{Boo}_\mathbf{d}$ in ab BD to the morphism and $\mathsf{ob}_{\mathsf{ab}}$ $\mathbf{r} \mathbf{D} \rightarrow \mathbf{BD}$ of distribution is universal and \mathbf{D} κ p op $\sum_{\mathbf{s}}^2$ is $\sum_{\mathbf{s}}^2$ in Boo_{lean} and $\sum_{\mathbf{s}}^2$ in \mathbf{t}_r^2 D \rightarrow B $\sum_{\mathbf{s}}^2$ op_{is} o_{is} distributive lattices, the Boo_lean homomor p_{vs}

2.3 Semilattices

 \blacksquare require \mathbb{R} and some sort, we are dealing with a semi-finite limit and \mathbb{C} there are two sorts of semilattice from the order-theoretic point of view. A meet semilattice $\sum_{i,s}$ partially ordered set in which every finite subset in which every finite subset $\sum_{i,s}$ A join semilattice $\frac{1}{n}$ partially ordered set in $\frac{1}{n}$ is $\frac{1}{n}$ has a meeting separate necessarily has a distinguished top element, \mathbf{u} and \mathbf{u} \mathbf{v} \mathbf{s} a \mathbf{s} join \mathbf{s} and \mathbf{s} a A morphism of meet semilattices is a map which preserves finite meets \blacksquare op_{is} of on_{is} \blacksquare morphism preserves in $\text{dom}_{\mathcal{B}}$ $\mathbf{p}_{\mathbf{r}\mathbf{s}}\mathbf{n}$ maps $\mathbf{p}_{\mathbf{r}\mathbf{s}}$ or order. Notice that more semi-

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 T_{ref} the first three define a computation of T_{ref} is T_{ref} or \mathbf{p}_{ref} $\mathbf{y} = \mathbf{n} \Rightarrow \mathbf{n}$ is identical in the order rela- $\sum_{n=0}^{\infty} \frac{n}{n}$ and $\sum_{n=0}^{\infty} \frac{n}{n}$ and $\sum_{n=0}^{\infty} \frac{n}{n}$ and $\sum_{n=0}^{\infty} \frac{n}{n}$ $a < b$ a $\vee b$ b. A meet semilate satisfies the same equations if λ replaces λ replaces λ replaces λ n r_s order relation must be recovered in the opposite senses in the opposite senses: $a \leq b$ a ∧ b a. $\begin{array}{ccccccc}\n0 & & & & \end{array}$ pono $\begin{array}{ccccccc}\n\text{so} & & \text{n} & & \text{n} & \end{array}$ \blacksquare if $\lceil \ln \frac{1}{n} \rceil$ is a matter of terminology corresponding to the terminology c the distinction between and an additive and a multiplicative group. The distinction \mathbf{p}_eff are distinction and a multiplicative group. on only arises the ordering one of the two senses for the two senses for the two senses for the order two senses for the two senses for the order A $\frac{1}{15}$ meet is said to be complete is subset has $\frac{1}{15}$ if $\frac{1}{15}$ n \mathbf{s} decree is complete. Dually a join semilation semi- $\frac{1}{2}$ every subset has a $\frac{1}{2}$ of complete $\frac{1}{2}$ of complete $\frac{1}{2}$ on and complete meeting $\frac{1}{2}$ $\frac{1}{15}$ are particularly rich in structure. For $\frac{1}{15}$ and Tierney $\frac{1}{15}$ $\frac{1}{\sqrt{2}}$ call the categories of suplattices $\frac{1}{\sqrt{2}}$ n_{rss arises} principal that each isomorphic to its isomorphic to its opposite that each isomorphic to its opposite that each isomorphic to its opposite to i rS \blacksquare A $_{\mathsf{R}}$ periodice R and R is \blacksquare arbitrary meeting as well as a subset of $\lim_{n \to \infty} \frac{1}{n} \frac{1}{n} \int_{\mathbb{R}} \frac{1}{n} \$ \int ono_{ts} $\frac{1}{2}$ on $\frac{1}{2}$ \wedge S $\qquad \vee \{a \in A | S \subseteq \uparrow a\}.$ μ n n is $\int \text{O}_\mathfrak{B}^3$ is $\frac{n}{\mathfrak{B}}$ in A is \mathbb{R} partially ordered set with a partially $\int \mathcal{O} \, \mathbb{n}_{\mathcal{S}}$ with a set with arbitrary set with a se oppo_{site} p_{artial} A° is a superior $\lim_{n \to \infty} \frac{p_n}{n}$ is $\lim_{n \to \infty} \frac{p_n}{n}$ \mathcal{L}_{NS} the metally the opposite of an inflattice is an inflattice is an inflattice is an inflattice. Now inflattice is an inflattice. Now inflatti **let A** and B $\underset{\kappa}{\kappa}$ P_a m_a $\underset{\kappa}{\kappa}$ and f_{κ} A \rightarrow B be a p_{eris}on $\underset{\kappa}{\kappa}$ p_a $\underset{\kappa}{\kappa}$ $\mathcal{S}_{\mathcal{S}}$ f p_{reserve}s arbitrary join_s in the p_{reserve} or $\mathcal{S}_{\mathcal{S}}$ unique morphism or $\mathcal{S}_{\mathcal{S}}$ or p_{artial} order sets f : B \rightarrow A $\frac{1}{15}$ is n condition $f \rightarrow g \leq b$ $a \leq f \rightarrow b$ o and $\mathbf{A} = \mathbf{A} \times \mathbf{B}$ and $\mathbf{B} = \mathbf{B}$ is given by $\mathbf{B} = \mathbf{B} \times \mathbf{B}$ and $\mathbf{B} = \mathbf{B} \times \mathbf{B}$ $f \downarrow \nightharpoonup$ $\forall \{a \in A | f \downarrow a \leq b\}.$ f p_{reserve}s arbitrary meets since A f on_{re} in f p_{reserve}s them. $n \cdot n$ g: $f: A \rightarrow B$ and $g: B \rightarrow A$ are order-preserving maps between p_{∞} maps between p_{∞} maps between p_{∞} p \bullet 0 \bullet \bullet \bullet \bullet \bullet \bullet \bullet f_{\prec} a \leq b $a \leq g_{\prec}$ b

 3 Thus, as an object, a complete semilattice or either sort is in fact a complete lattice. However, since a morphism of suplattices need not preserve meets, nor a morphism of 4848-1274252((()-0i)0.74**2512d31dt(n)difis,@2**283(2056]TJ 83(20131(,)-444.923(s)-3.0.26064(r)-0.649399(y)-396.625(j)-1.94207(o)-2.26269

for all a ∈ A and b ∈ B, we refer to f as the left adjoint of g and g as the right adjoint of f. The left adjoint preserves joins and the right adjoint preserves meets. Either uniquely determines the other by the condition f(a) = V {b ∈ B|a ≤ g(b)} g(b) = W {a ∈ A|f(a) ≤ b}. Now meets in A are precisely the joins of A^o so that f can equally be considered as a morphism of suplattices f o : B^o → A^o . This establishes a bijection between the suplattice morphisms from A to B and the suplattice morphisms from B^o to A o . In fact the functor which sends a suplattice to its opposite and a morphism to its right adjoint establishes an isomorphism between the category of suplattices and its opposite. Dually the functor which sends an inflattice to its opposite and a morphism to its left adjoint establishes an isomorphism between the category of inflattices and its opposite. From an algebraic point of view our concern in this paper is with probability measures on complete semilattices in general. Our immediate practical concern, however, is with probability measures on finite semilattices. To avoid complications, we shall therefore restrict attention to the full subcategories of finite suplattices and finite inflattices.

2.4 Probability Measures on Inflattices

Definition 1 A $\rm{p~o}$ and a finite inflattice A is a real unitinterval valued function $\mathbf{p} \mathbf{A} \rightarrow \mathbf{q}$, satisfying

$$
p_{\gamma}^{\mathbf{b}}\text{ with } \mathbf{a}_{\gamma}^{\mathbf{b}} \text{ with } \mathbf{b}_{\gamma}^{\mathbf{b}} \text{ with } \mathbf{b}_{\
$$

 $a \circ A \circ \downarrow_{\mathfrak{F}}$ DA

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for all $a \in A$. Moreover this function, called the α_{max} of p, is unique when it exists.

It is easy to see that a probability measure p and its dual p ^o have the same density. Thus if m is the density of p then p o is given by p o (a) = X{m(b)|a ≤ b}. Using the representation in terms of densities it is easy in the finite case to establish an important result concerning measures on direct sums of inflattices. Let A and B be inflattices. Denote by A ⊕ B the set of all pairs (a, b) with a ∈ A and b ∈ B. Then A ⊕ B is an inflattice under the coordinatewise partial order. It is in fact the biproduct of A and B in the category of inflattices under the obvious injections and projections

Proposition 6 If p and q are probability measures on the finite inflattices A and B respectively, then the function $p \times q$ defined for all $a \in A$ and $b \in B$ by

 \mathbb{R} p \times q \mathbb{R} a, b) = p \mathbb{R} q \mathbb{R} q \mathbb{R}

is a probability measure on $A \oplus B$.

Proof \mathbf{n}_{∞} **m** of $\mathbf{p} \times \mathbf{q}_{\infty}$ is point is a product of the product o n_{res} and p and res m_p is the m_p is the density of res of res of res p $\overline{\mathsf{n}}$ $\overline{\mathsf{m}}$ $\overline{\mathsf{m}}$ $\overline{\mathsf{n}}$ $\overline{\mathsf{n}}$ $\overline{\mathsf{n}}$ $\overline{\mathsf{n}}$ $\overline{\mathsf{n}}$ $\overline{\mathsf{n}}$ $\overline{\mathsf{n}}$

Corollary 7 If p and q are probability measures on an inflattice A then the function $\mathbf{p} \cdot \mathbf{q}$ defined for all $\mathbf{a} \in \mathbf{A}$ by

 \prec p \cdot g \prec a) p \prec a) q \prec a)

is also a probability measure on A.

Proof Let ∆: A → A ⊕ A be the diagonal morphism sending a to (a, a). Then p · q = (p × q) ◦ ∆ which is a probability measure on A by Lemma 2. ✷ Corollary 7 allows us to introduce a binary operation on probability measures on an inflattice which forms the basis of Dempster's rule of combination in the theory of belief functions. Let Pr (A) denote the set of probability measures on an inflattice A and let the binary operation be defined by p ⋆ q = (p o · q o o for all p, q ∈ Pr (A).

Proposition 8 Pr \mathbf{A} is a commutative monoid under .

2.4 Probability Measures on Inflattices

Proof Associativity follows from associativity of real multiplication and the fact that p oo p and commutativity is obvious. The unit of the monoid is the measure whose dual has the constant value 1. ✷ The unit v of Pr (A), which we call the vacuous measure on A, is given explicitly by ^v(a) = (1 if a = 1 0 otherwise. It is easy to show that if p has density m^p and q has density m^q then p ⋆ q has density⁵ m(a) = X{mp(b)mq(c)|a b ∧ c}. Now let f: A → B be an inflattice morphism and let f ^o denote its left adjoint as a morphism of the opposites B^o and A^o considered as inflattices. Then we define Pr (f): Pr (A) → Pr (B) by Pr (f)(p) = (p ^o ◦ f o) o for all p ∈ Pr (A).

Proposition 9 Pr is a (covariant) functor from the category of finite inflattices to the category of commutative monoids.

Proof $p \in Pr_{\prec}A$ nd $f_{\prec}A \rightarrow B$ is an inflattion operation $f^0 \rightarrow A^0$ is a integration of p is an integration of p is a integration $\mathsf{p}^{\mathsf{o}}\circ\mathsf{f}^{\mathsf{o}}\in\mathsf{Pr}_\prec\mathsf{B}^{\mathsf{o}}_{\mathsf{S}}\quad\mathsf{p}^{\mathsf{o}}\in\mathsf{Pr}_\prec\mathsf{A}^{\mathsf{o}}_{\mathsf{S}}\qquad\mathsf{R}^{\mathsf{o}}\circ\mathsf{f}^{\mathsf{o}}_{\mathsf{S}}\circ\mathsf{P}\mathsf{r}_\prec\mathsf{f}_\prec\mathsf{B}\in\mathsf{Pr}_\prec\mathsf{B}_\prec$ o _{ns} ppo_{rs} **p**, **q** ∈ Pr_{\lt}A₂ in $\mathsf{Pr}_\prec \mathsf{f}_\succ \mathsf{p} \quad \mathsf{q} \qquad \mathsf{p} \quad \mathsf{q} \circ \circ \mathsf{f}_\succ^\circ \circ$ \sim p^o · q^o of $\sqrt{$ ^o $\overline{}$ $\overline{}$ (p^o o **f**) $\overline{}$ o **f**) $\overline{}$ o **f**) $\overline{}$ $\mathbb{E}_{\mathbb{P}^0} \circ \mathsf{f}^0_\lambda \circ \mathbb{P}^0 \circ \mathsf{f}^0_\lambda \circ$ $Pr_{\prec} f_{\gamma} \prec p$ $Pr_{\prec} f_{\gamma} \prec q$ n Pr (f) even the unit of Pr (A). The unit of Pr (f) is a monomed vector of P $\overline{\mathsf{O}}$ oop_{rs} $\overline{\mathsf{O}}$ _{rs} $\overline{\mathsf{P}}$ **F**₋ $\overline{\mathsf{g}}$ o $\overline{\mathsf{P}}$ **r**_{- $\overline{\mathsf{g}}$ o $\overline{\mathsf{P}}$ **F**_{- $\overline{\mathsf{g}}$ of $\overline{\mathsf{P}}$ of $\overline{\mathsf{p}}$ in $\overline{\mathsf{p}}$}} op_{ism}s $f \colon A \to B$ and $g \colon B \to C$ on \mathbf{m} and \mathbf{m} and $\mathbf{g} \circ f \circ g$ $f^{\circ} \circ g^{\circ}$ n Pr en event in event in p events in events in \Box $p o$ a noPr $\frac{1}{15}$ is $p a$ is $p o n n$ is $q o$ in terms of densities. \mathbf{p}_{∞} po \mathbf{A} with density measurement density measurement \mathbf{p}_{∞} morphism on is n infor Pr (f) is no \mathbf{B} is \mathbf{B} is $\mathbf{b} \in \mathbf{B}$ m_{f} (b) $\sum \{m_{\alpha}a|f_{\alpha}a\}$ b).

3.1 Uniform Measures

Lemma 11 Let f be any real-valued function on a finite inflattice A with $n \nmid$ ranks. Then there exists a proper probability measure p on A and a sequence of positive real number K_0, \ldots, K_n such that for each i \ldots, n

$$
\textbf{p}^{\text{o}} \textbf{q} \qquad \textbf{K}_i \quad \text{p} \textbf{f} \textbf{q}
$$

whenever $n \sqrt{a}$ i.

Proof A₃₅
$$
\begin{array}{c} \n\pi \text{ on } \mathfrak{m}_{i} \xrightarrow[\mathfrak{F}]{} \n\end{array}
$$
 on $\{\mathbf{a} \in \mathbb{A} \mid \begin{array}{c} \n\pi \text{ on } \mathfrak{m}_{i} \xrightarrow[\mathfrak{F}]{} \n\end{array}$ on $\{\mathbf{a} \in \mathbb{A} \mid \begin{array}{c} \n\pi \text{ on } \mathfrak{g} \leq \mathbf{i} \} \n\end{array}$

$$
g_{i \sim \frac{a}{2}} \qquad \sum \{m_{i-1 \sim \frac{b}{2}} \mid a < b\}
$$
\n
$$
k_i \qquad n \quad \frac{p \ f_{\sim \frac{a}{2}}}{g_{i \sim \frac{a}{2}}}
$$

n infimum taken over ${a \in A}$ is the induction by induction ${a \in A}$, in the induction by induction by induction by induction by induction ${a \in A}$ in ${a \in A}$, in the induction by induction by induction by induction by i on_{st} n \mathbf{k}_i is n and \mathbf{p}_i and \mathbf{p}_i and \mathbf{m}_i and \mathbf{m}_i $\sum_{n\leq n}$ non negative. In particular m_n is non-negative. Let m m_n in o n on

$$
\sum \{m_{i\sim} \phi \mid a \leq b\} \quad \text{if} \quad p \mathbf{f}_{\prec} \mathbf{g}
$$
\n
$$
\sum_{a} \frac{n}{A} m_{\prec} \mathbf{g} \quad \mathbf{i}_{\neg s} n \quad a < b \quad n \quad n \prec \mathbf{g} \quad \mathbf{i} \quad p_{\prec} \quad n \prec \mathbf{g} < \mathbf{i} \quad \mathbf{g}
$$

$$
\begin{array}{ccccccccc}\n\mathbf{P} & \mathbf{s} & \mathbf{p} & \mathbf{m} & \mathbf{m} & \mathbf{s} & \mathbf{p} & \mathbf{p} & \mathbf{p} & \mathbf{s} & \mathbf{s} \\
\mathbf{p} & \mathbf{p} & \mathbf{s} & \mathbf{p} & \mathbf{s} & \mathbf{s} & \mathbf{m} & \mathbf{s} & \mathbf{p} & \mathbf{p} & \mathbf{p} & \mathbf{p} & \mathbf{s} \\
\mathbf{p} & \mathbf{p} \\
\mathbf{p} & \mathbf{p} \\
\mathbf{p} & \mathbf{p} \\
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\mathbf{p} & \mathbf{p} \\
\mathbf{p} & \mathbf{p} \\
\mathbf{p} & \mathbf
$$

Definition 3 If f is any real-valued function on a finite inflattice A we denote by \therefore f the proper probability measure defined by the above construction.

Proposition 12 Pr_<A₂ /Un_<A₂ is an Abelian group.

Proof ppo_{rs} **p** pn_{rs} o Pr (A). q . Let $\text{p}^{\circ}_{\text{s}}$ by np \uparrow a^o p opo on of o p_{rop}ortional to p_{rop}ortions in the new section of p° and p° an $\mathbf{n} \rightarrow \mathbf{p}$ q

Pr \mathbf{P} roof $\mathsf{p} \equiv \mathsf{q}$ there are uniform p v such that v v such that p u q v , n a p^o−a q^o a v^o−a u^o po s ans oN₂A, n s o s $p \circ o \circ \phi$ is the only induction on $p \circ o \circ \phi$ in on o \mathbf{n} on reg in Lemma 11. We leave the interest of the interest reader. (3) is a restatement of the definition of uniformity and (4) is a restatement $\begin{array}{ccc} 0 & & & D \\ 1 & \bullet & & D \end{array}$

Proposition 14

 $Pr_{\alpha}A$ /Un $_{\alpha}A$ is isomorphic to the additive group of $L_{\alpha}A$ /N $_{\alpha}A$.

$$
\text{Proof} \qquad \text{in} \quad \text{and} \quad \text{Pr}_{\text{in}} \mathbf{A} \text{ and } \text{Pr}_{\text{in}} \mathbf{A} \implies \text{L}_{\text{in}} \mathbf{A} \text{ and } \text{Pr}_{\text{in}} \mathbf{A} \text{ are } \text{Pr}_{\text{in}} \mathbf{A} \text{ and } \text{Pr}_{\text{in}} \mathbf{A} \text{ are } \text{Pr}_{\text{in}} \mathbf{A} \text{ and } \text{Pr}_{\text{in}} \mathbf{A} \text{ are } \text{Pr}_{\text{in}} \mathbf{A} \text{ and } \text{Pr}_{\text{in}} \mathbf{A} \text{ are } \text{Pr}_{\text{in}} \mathbf{A} \text{ and } \text{Pr}_{\text{in}} \mathbf{A} \text{ are } \text{Pr}_{\text{in}} \mathbf{A} \text{ and } \text{Pr}_{\text{in}} \mathbf{A} \text{ are } \text{Pr}_{\text{in}} \mathbf{A} \text{ and } \text{Pr}_{\text{in}} \mathbf{A} \text{ are } \text{Pr}_{\text{in}} \mathbf{A} \text{ and } \text{Pr}_{\text{in}} \mathbf{A} \text{ are } \text{Pr}_{\text{in}} \mathbf{A} \text{ and } \text{Pr}_{\text{in}} \mathbf{A} \text{ are } \text{Pr}_{\text{in}} \mathbf{A} \text{ and } \text{Pr}_{\text{in}} \mathbf{A} \text{ are } \text{Pr}_{\text{in}} \mathbf{A} \text{ and } \text{Pr}_{\text{in}} \mathbf{A} \text{ are } \text{Pr}_{\text{in}} \mathbf{A} \text{ and } \text{Pr}_{\text{in}} \mathbf{A} \text{ are } \text{Pr}_{\text{in}} \mathbf{A} \text{ and } \text{Pr}_{\text{in}} \mathbf{A} \text{ are } \text{Pr}_{\text{in}} \mathbf{A} \text{ and } \text{Pr}_{\text{in}} \mathbf{A} \text{ are } \text{Pr}_{\text{in}} \mathbf{A} \text{ and } \text{Pr}_{\text{in}} \mathbf{A} \text{ are } \text{Pr}_{\text{in}} \mathbf{A} \text{ and } \text{Pr}_{\text{in}} \mathbf{A} \text{ are } \text{Pr}_{\text{in}} \mathbf{A} \text{ and } \text{Pr}_{
$$

([p]) = [log p o and : L(A)/N(A) → Pr (A)/Un(A) by ([f]) = [reg f . These are well-defined in view of (1) and (2) above. Now (3) implies that ([p]) = [0] iff [p] = [0] and ([p] + [q]) = ([p ⋆ q]) = [log (p ⋆ q) o = [log p ^o + log q o = [log p o] + [log q o ([p]) + ([q]).

 T_{max} is a group θ and θ in θ and θ function. Now θ function. Now θ \mathcal{S}_{NS} ppo \mathcal{S}_{NS} . **f** \mathcal{S}_{max} o **L**₍**A**). Then

(([f]) = ([reg f]) = [log (reg f) o] = [f

 $\begin{array}{rcl}\n\mathcal{A} & \mathcal{A} & \mathcal{A} & \mathcal{A} \\
\mathcal{A} & \mathcal{A} & \$ \overrightarrow{O} _{is} \overrightarrow{O} _{is} in on Pr (A)/Un(A). In \overrightarrow{S} is \overrightarrow{O} p homomorphism it follows that is also a group homomorphism and we are \overline{O} through \Box

3.2 Regular Measures

op \mathbf{p} po_{rses} is non n n o deal \mathbf{p} and \mathbf{p} measures. We show that the show next that each extension of \mathbf{n}_{res} and \mathbf{n}_{res} and \mathbf{n}_{res} \blacksquare and \blacksquare and \blacksquare as a canonical representative. The canonical representative as a canonical representative. The canonical representative of P is P

Definition 4 Let . Pr A \rightarrow Pr A be defined by

 $\mathbf{p} \in \mathbf{p}$ and \mathbf{p}^0 .

We say that a proper measure p is regular if and only if $\Box p$ = p and we denote by Reg_< A the set of regular measures on a finite inflattice A.

$$
\begin{array}{cccc}\n\text{no} & \text{so} & \text{a} & \text{no} & \text{Pr}_{\prec} \mathbf{A} \\
\text{no} & \text{Pr}_{\prec} \mathbf{A} & \text{on} & \text{P}_{\mathbf{A}} \\
\text{no} & \text{P}_{\mathbf{A}} & \text{P}_{\mathbf{A}} \\
\text{no} & \text{P}_{\mathbf{A}} & \text{P}_{\mathbf{A}} \\
\text{on} & \text{P}_{\mathbf
$$

Lemma 15 is idempotent: ○ . Hence $\Box \varrho$ is regular for all $p \in$ $Pr_{\prec} \mathbf{A}$.

Proof
$$
\bullet
$$
 f \bullet **p**⁰ $n \bullet$ **y** 0 **h** $p \circ \bullet$ **i** \bullet **j** \bullet **l** \bullet **j** \bullet **k** \bullet **u** $\$

Proposition 16 Each element of $Pr_{\prec} A$ /Un $_{\prec} A$ contains one and only one regular measure.

Proof Suppose [p] belongs to Pr (A)/Un(A). Then (([p])) = [p] implies that [reg (log p o)] = [p] or in other words [(p)] = [p]. But (p) is regular by Lemma 15. Hence every element of Pr (A)/Un(A) contains at least one regular measure. Now suppose that p ≡ q. Then (p) = (q) by (1) and (2) of Lemma 13. So if p and q are both regular we have p (p) = (q) = q Thus each equivalence class in Pr (A)/Un(A) contains at most one regular measure. ✷ This result means that Reg(A) is 6269(r)-402.135(29(S)1.94718(o.(e)3.56748(g)-2.2626wh)1.9482(6(s)-426.833()1.9482(7(c)1.878269(n)1.9482(d)i)09.9484]TJ

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3.4 Covariant Transformations \longleftarrow

$$
\begin{array}{llll}\n\text{or} & \mathbf{a} \in \mathbf{A} & \text{if} &
$$

is is and the condition of \mathbf{B} is in the condition \mathbf{B} is in the quotient B is in the $\text{Re}\left(z\right)$ and the same as it is it is for sets in the numerator $\text{Re}\left(z\right)$ n \triangleleft equality of $\frac{1}{\sqrt{2}}$ \mathbb{R} or \mathbb{R} be some fixed element of a finite inflation fin \mathbf{B} \downarrow **b** n o n_{ats} on o pon n o_{rr} p_a is $\begin{array}{ccccccc}\n\text{o} & \text{o} & \text{n} & \text{p} \Box \uparrow & \wedge \mathbf{b} & \mathbf{A} \rightarrow \mathbf{B} & & \text{s} & \text{n} & \mathbf{A} & \text{n} & \mathbf{a} \in \mathbf{A} & \text{o}\n\end{array}$ $a \wedge b \in B$ is point be pointed out, hence \downarrow , \downarrow need not in general behavior. a notice A in the category rILf $\mathbf{f}_{\mathbf{x}}$ in $\mathbf{f}_{\mathbf{x}}$ inclusion of $\mathbf{f}_{\mathbf{x}}$ in $\mathbf{f}_{\mathbf{x}}$ n nop_{res} a on B_{rs} spo_{ws i}s oppo_{is} besides power sets (or the rank opposite o is no \mathbb{R}^n and \mathbb{R}^n and there is a significant case where the significant case \mathbb{R}^n always occurs, namely \mathbb{R}^n and \mathbb{R}^n and \mathbb{R}^n are the significant case of \mathbb{R}^n and \mathbb{R}^n when A is the inflattion A is that a finite inflattice inflattice inflattice \uparrow , \uparrow \Box and order \Box b \Box b = 0. Then the inclusion of \Box inclusion of \Box inclusion of \Box \downarrow **b** n **A** always preserves also n \mathbb{P}_{\bullet} is \mathbb{P}_{\bullet} and \mathbb{P}_{\bullet} and \mathbb{P}_{\bullet} are \mathbb{P}_{\bullet} are \mathbb{P}_{\bullet} and in Boolean po and $\lim_{n \to \infty} \sin \frac{1}{n}$ in the constant $\lim_{n \to \infty} \frac{1}{n}$ in the case where $\lim_{n \to \infty} \frac{1}{n}$ ono ι o**A** Appen Reg onop_{hism} \wedge b**A** → B yields \mathcal{A}^n and $n_{\mathcal{R}}$ corresponding $\mathsf{Reg}_\prec \mathsf{A}$ or $\mathsf{Reg}_\prec \mathsf{B}$ corresponding to positive on on $\mathcal{A}_{\mathcal{R}}$ on on $\mathcal{A}_{\mathcal{R}}$ and $\mathcal{A}_{\mathcal{R}}$ and $\mathcal{A}_{\mathcal{R}}$ and $\mathcal{A}_{\mathcal{R}}$ and $\mathcal{A}_{\mathcal{R}}$ on A

3.5 Contravariant Transformations \downarrow

Pr_{\mathbf{x}} p_{reserve}s one operation, because $\mathbf{p}_{\mathbf{x}}$ and $\mathbf{p}_{\mathbf{x}}$ or nd a position component compone n on only consider the $p \rightarrow \sim \mathbf{b}$ when \mathbf{b} / Co $p \circ_{\mathbf{r} \cdot \mathbf{s}} n$. \mathbf{r}_s will also unit of Pr (\mathbf{p}_s) not to the unit of \mathbf{p}_s but to a measure to on n on $\mathbf{b} \times \mathbf{n}$ on in pop one). Can improper one n_{res} s o o p po_{rses} non no n no n dono ppo_{rs} o p o_{rs} ϵ respectively about p directly about p direc onot on on $\frac{1}{15}$ due to a drug. But it does suggest that it does not a drug. But it does not a drug. Support it as a drug. It does not a drug is involved, it may as well be derived, it may be derived, it may be derived as A as drug B. In plansion $\mathbf{p}_{\mathbf{r},\mathbf{s}}\mathbf{n}$ on of $\mathbf{p}_{\mathbf{r},\mathbf{s}}$ and the vacuus measure, where $\mathbf{p}_{\mathbf{r},\mathbf{s}}$ and the vacuus measure, where $\mathbf{p}_{\mathbf{r},\mathbf{s}}$ and $\mathbf{p}_{\mathbf{r},\mathbf{s}}$ and $\mathbf{p}_{\mathbf{r},\mathbf{s}}$ and $\mathbf{p}_{\mathbf{r},\mathbf{s}}$ on on the numerator.

 \top

{subject drug} {something else}

⊥

\mathcal{L} A simple alternative.

neutral notation has been used to top and both $\mathbf{r}_\mathbf{S}$ and both $\mathbf{r}_\mathbf{S}$ as not to $p \downarrow \rightarrow$ or $q \downarrow 0$ in \mathcal{A} on \mathcal{A} or \mathcal{A} or \mathcal{A} in \mathcal{A} or \mathcal{A} in \mathcal{A} or \mathcal{A} in $\mathcal{$ a superior even a Boo_{lea}n algebra a superior the appropriate the appropriat no on o po \bullet $\sum_{\mathbf{r},\mathbf{s}}$ $\sum_{\mathbf{S}}$ $\sum_{\mathbf{S}}$ particle \mathbb{R} is the set is to have a semi-method set is to have a semi-method is \mathbb{R} in the \mathbb{R} n_{vs} on o_{ts} n_{sss}o_{ts} of cause in n_{vs} when n_{vs} **by necessary** on on o sufficient on and α or a sufficient condition. P_{\blacktriangleleft} on \mathcal{E} on \mathcal{E} on \mathcal{E} on \mathcal{E} on \mathcal{E} and event: $n \rightarrow \infty$ not be even the patient motion $\mathbf{r}_\mathbf{S}$ drug. $\begin{array}{ccc} \hbox{m on} & \hbox{on} & \mathbf{0} \end{array}$

In the first case taking the first case of α necessary condition for the drug is viewed as a necessary condition for the α in B $_{\rm rs}$ not satisfy necessary condition. It is not parameter on the condition. It is not parameter on $_{\rm rs}$ and $_{\rm ns}$ condition. It is not parameter on the condition. It is not parameter on $_{\rm rs}$ and $_{\rm ns}$ co $p \rightarrow \text{p}$ the event would not have part if the patient is patient if the patient is patient if the patient in the p $\mathbf{h}_{\mathbf{s}}$ o nop $\mathbf{p}_{\mathbf{s}}$ or if the prediction is not been taking the patient had not been taking \mathbf{p} α drug at the same time α drug interaction, α and α mean by mean $\mathcal{L}(\mathbf{r}) = \mathbf{r} \cdot \mathbf{r}$ is that $\mathbf{r} \cdot \mathbf{r}$ is the drug had something to do with it is the drug had something to do with it is $\mathbf{r} \cdot \mathbf{r}$ and $\mathbf{r} \cdot \mathbf{r}$ is the drug had something to do with it is $\$ the drug was a particle drug is drug is drug is held to decree taking the drug is $\frac{1}{\sqrt{2}}$ be a sufficient condition for the event. The event of the event of the event of \mathbb{R} that α is neglector to cause the taking of the drug. The drug of the drug of the drug of the drug. The drug of the drug of the drug. The drug of the drug. This se pure n_{max} is the means electron of means electron \mathbf{u} above that \mathbf{u} is that \mathbf{u} is the \mathbf{u} n $\frac{1}{100}$ the speaking is uncertainty in the speaking is used when we say the speaking is used when we say the speaking is used when we say that n is used when $\frac{1}{100}$ is used when $\frac{1}{100}$ is used when \frac $\mathcal{O}_{\blacktriangleleft}$ the drug of the drug agree, the drug agree, to take and most would agree. The drug agree, to take and most would agree. extreme example, the is a cause of death in $\frac{1}{10}$ is a The same dual interpretations must be given to the alternative hyperpretation of \mathbf{m} po_{thesis} \mathcal{P} in the first cause \mathcal{P} is means the first case it means the first case it means that \mathcal{P} \mathcal{S} o on on \mathcal{S} partial cause of the second case it the second case it the second case it is \mathcal{S} $m_{\mathbf{r}} = \mathbf{r} \mathbf{\Theta}$ of one one $\mathbf{r} \mathbf{s} = \mathbf{r} \mathbf{\Theta}$ the event of the event of the event.

Proposition 21 Every probability measure on a finite suplattice A has a

n op on $o_{\mathfrak{H}}$ n $_{\mathfrak{B}}$ n $_{\mathfrak{B}}$

$$
\mathbf{v}_{\mathbf{y}} \mathbf{v}_{\mathbf{y}}
$$

6.2 Covariant Transformations

power set suplattice PX, for example, with S a subset of X and |X| n rank(S) is just the cardinality of S with rank(X) = n and rank(∅) = 0. In

Example 1 Let A be a finite suplattice. Then W : PA → A, which exhibits $A_{\mathbf{r}, \mathbf{s}}$ on o $\mathbf{r}, \mathbf{p}, \mathbf{q}$ is an epimorphism in rSLf. Its right $\int \rho \sin \theta \sin \theta$ is the down-segment mapping. Example 2 \blacksquare S and \blacksquare a finite a finite supplement supplement supplement and let B S∪{ } $\begin{array}{llll}\n\text{non o } \mathsf{S} & \text{op}_{\mathsf{A}} & \text{n o } \mathsf{A} & \text{n f}_\mathsf{P} \mathsf{A} \rightarrow \mathsf{B} & \text{n} \\
\end{array}$ f_{\prec} a $\begin{cases} a & a \in S \\ 0 & a \end{cases}$ $\begin{matrix} 1 & 0 \\ 0 & 0 \end{matrix}$ is the set of \mathbf{S} . is o n p n rSLf. Its i on is the index is the index is the inclusion. **Example 3** ppo_{rs} **A** PX is a finite power set supple pop_{rs} is power set order $n_{\mathcal{A}, \mathcal{S}}$ onn Y _{is is} o**X.** B {S ⊆ X|Y ⊆ S} θ collection of all supersets of θ is θ is θ is $n \in \mathbb{N}$ intersections, namely \mathbf{A} , and \mathbf{A} and \mathbf{A} and \mathbf{A} are corresponding to \mathbf{A} and \mathbf{A} are \mathbf{A} on p_{rsit} \cup Y_{'s} A \rightarrow B w_{hich} senses S ⊆ X o_{ns} non

Y $\underset{\mathbf{x}}{\cdot}$ \downarrow on $\underset{\mathbf{x}}{\cdot}$ and $\underset{\mathbf{x}}{\cdot}$ and $\underset{\mathbf{x}}{\cdot}$ subsets and the inclusion of $\underset{\mathbf{x}}{\cdot}$ is $\underset{\mathbf{x}}{\cdot}$ is $\underset{\mathbf{x}}{\cdot}$ is $\underset{\mathbf{x}}{\cdot}$ is $\underset{\mathbf{x}}{\cdot}$ equality of $\mathbf{r}_{\mathbf{s}}$ include the supersets of $\mathbf{s}_{\mathbf{s}}$ in $\mathbf{s}_{\mathbf{s}}$ in $\mathbf{s}_{\mathbf{s}}$ in $\mathbf{s}_{\mathbf{s}}$

6.3 Contravariant Transformations

 \approx discussion here \approx \approx discussion of contractly the discussion of contractly the discussion of contractly transformation of contractly transformation of contractly transformation of contractly transformation of co on_s opo \bullet is son inflattices. Again we see the immediate inflaton inflattices. Again we see the immediate inflattices. Again we see the immediate inflattices. Again we see the immediate inflattices. Again we see the p \bullet s \bullet s $n_{\mathbf{s}}$ of constraints in expressions in equations in equations in equation of conditional effect of conditional equations in equation of conditions in equation in equation of conditions in equation in e op n on_s. Con_s \cdots n \cdots is set \cdots do co_{ns} \cdots n \cdots on_{is} n inn inn p_{ress}n opinion \mathbf{p} and \mathbf{p} and \mathbf{p} and \mathbf{p} certain trait of \mathbf{p} and \mathbf{p} and \mathbf{p} non \blacksquare directly directly above. But it is the three Paris is likely to choose. But it is likely to choose $\frac{1}{\sqrt{2}}$ or is not going to choose $\frac{1}{\sqrt{2}}$ is $\frac{1}{\sqrt{2}}$ or $\frac{1}{$ $\overline{O_0}$ $\overline{O_1}$ $\overline{O_2}$ $\overline{O_3}$ $\overline{O_4}$ $\overline{O_5}$ $\overline{O_6}$ $\overline{O_7}$ $\overline{O_8}$ $\overline{O_8}$ $\overline{O_9}$ $\overline{O_9}$ $\overline{O_9}$ $\overline{O_9}$ $\overline{O_9}$ $\overline{O_9}$ $\overline{O_9}$ $\overline{O_9}$ $\overline{O_9}$ $\overline{O_9}$ p_{ress} on on a conditional proposition being false. In numerical proposition being false in numerical proposition and \mathbf{A} p_{w} expression of the pression on the pressure on the p_{w} measure on the quotient is pressure on the quotient in the p_{w} measurement is not the quotient in the quotient in the quotient is not the quotient \mathbb{R} both both element is the lifted to a mean behavior is the lifted to a mean behavior is the mean behavior in \mathbb{R} behavior in s on the \mathcal{Q} \mathcal{Q} one of \mathcal{Q} is \mathcal{Q} or \mathcal{Q} is \mathcal{Q} and \mathcal{Q} a positive condition is \mathcal{Q} $\overline{^{\dagger}}$

 $\mathbf{I}_\mathbf{s}$ omnon, in the addition theory, to speak as the independence the independence the independence in the independence of $\mathbf{I}_\mathbf{s}$ as the independence of $\mathbf{I}_\mathbf{s}$ as the independence of $\mathbf{I}_\mathbf{s}$ as on_{ts} a factual ϵ on the probability measure is the pro $n \times n$ is the question probability measure is propriate to a given probability measure is appropriate to a given probability measure is appropriate to a given probability measure is appropriate to a given probability meas is one is \blacksquare one ones at \mathcal{P} and the probability so, if we all agrees that probabilities are that probabilities are that probabilities are the probabilities are the probabilities are the probabilities are the pro o he determined by counting frequencies and if \mathbf{n} and if \mathbf{n} and if \mathbf{n} and \mathbf{n} $r_{\rm s}$ B not always $r_{\rm s}$ for $r_{\rm s}$ $\mathbf{r}_\mathbf{S}$ no algorithm for determining the correct probability measure and correct probability measure and $\mathbf{r}_\mathbf{S}$ in $\mathbf{r}_\mathbf{S}$ therefore no algorithm for deciding when $\mathbf{r}_\mathbf{S}$ are not two events are not two event npnn no \bigotimes po is nn \bigcup , n nn_s npnnn_s

 $\pmb{\gamma}$

- \bullet on_{rs} of the event in the particular patient was particular patient was particular patient was patien \overline{O} p drug relations \overline{O} $p \quad n \quad \frac{1}{5} \quad n \quad n \quad \frac{1}{5}$ lead to the event. \mathbb{R} because is a pharmacological reason when \mathbb{R} and \mathbb{R} reason when \mathbb{R} is a pharmacological reason when \mathbb{R} is a pharmacological reason when \mathbb{R} is a pharmacological reason when \mathbb{R} is r_s particular event. \mathbf{r} in occurrence of the event in normal property in practice in \mathbf{r} in normal property in \mathbf{p} in nop particular particula
- $\begin{array}{ccccccc}\n\text{o} & \text{no} & & \text{se} & \text{po} \\
\text{se} & \text{po} & & \text{me} & \text{me} & \text{me} & \text{po} & \text{me} &$ \blacksquare resolved when the drug was opped.
- \mathbb{R}^A are p_{res} it like \mathbb{R}^B and \mathbb{R}^A it is it like \mathbb{R}^A it is it like \mathbb{R}^A $\mathbb{R}^{\mathbf{B}}$ between it somewhere is in the interval between \mathbf{B} and \mathbf{B} T_{S} is no easy T_{S} are T_{S} are points to make T_{S} or T_{S} are points to make are points to make a response T_{S} or T_{S} are points to make T_{S} or T_{S} are points to make • the intuitive concept of independence in the independence in \mathbb{R} primitive independence in \mathbb{R} primitive in \mathbb{R} prim pooo onn nom • that there is so far no algorithm to substitute for individual judgment n n n o o \mathbf{s} o n n p n n
	- $\mathbf{p}_{\mathbf{r},\mathbf{s}}$ no $\mathbf{p}_{\mathbf{s}}$ in no $\mathbf{p}_{\mathbf{s}}$ and $\mathbf{p}_{\mathbf{s}}$ on $\mathbf{n}_{\mathbf{s}}$ in the $\mathbf{p}_{\mathbf{s}}$ B_{vs} n o
		- \blacksquare \blacks

8 Elicitation

⁴ 8 ELICITATION

view is based on the idea that two items of \mathbb{R}^n of \mathbb{R}^n of \mathbb{R}^n which weighs of which weighs \mathbb{R}^n and \mathbb{R}^n which we interference, one of which we interference, one of which we interference o p_{recisel} in \circ of \circ popo_{rs} on \circ a in the other of \circ of \circ which is a given degree and the other of which is a given degree and the other of which is a given degree and the other of \circ other of \circ othe weight the same degree of the s on on \mathbb{R}^0 with contrary evidence \mathbb{R} as forces us to deal with contrary evidence as forces \mathbb{R}^0 with contrary evidence as forces \mathbb{R}^0 as follows. If an item of event against a to degree weight a to degree s, it is it effect is expressed by the addition in the addition in the group of n regular measures in the group of regular measures o \mathbf{s} , $\mathbf{P}_{\mathbf{s}}$ is \mathbf{s} the opposite situation in which is $\mathbf{P}_{\mathbf{s}}$ is \mathbf{s} on in $n \longrightarrow sp \longrightarrow m$ in the same absolute numerical in favour of a to the same absolute numerical in the same absolute numerical indicates p degree.

where $\mathbf{a}_{\mathbf{g}}$ is a is a is $\mathbf{a}_{\mathbf{g}}$ in the interval interval is not interval in \mathbf{a} $0 \rightarrow s$ ppo s that an item of evidence weight precisely in favour of an item of an item s element a to a given degree strategy of the additional behavior in the additional by the addition in the addition in the addition of the addition of the addition in the addition in the addition of the addition in the addi in the group of $\prec_{\mathcal{B}} P \blacktriangleleft \rightarrow$ regular $\sim_{\mathcal{B}} P \blacktriangleleft \rightarrow$ the measure corresponding corresponding to the measure corresponding to the measure corresponding to the measure corresponding to the measure corresponding to \overline{O} evidence and \overline{R} to the same degree. $\frac{1}{15}$ are some examples. Suppose a $\frac{1}{15}$ and $\frac{1}{15}$ are $\frac{1}{15}$ and $\frac{1}{15}$ are o_f P and the that the that the strength P against his control of strength P against his control venus. The strength P against his control venus. The strength P against P against P against P against P a T_{S} T_{S} of Table 5. On the other hand evidence in favour of the other hand evidence in favour of the other hand evidence in favour of the T_{S} ('Juno or Minerva or Venus', 1) ('Juno or Minerva', 1) ('Juno or Venus', 1) ('Minerva or Venus', 1) ('Juno', 1) ('Minerva', 1) ('Venus', 0.6) ('', 0) \blacktriangleright \blacktriangleright Evidence against Var ₁₅. \mathbf{z} and \mathbf{z} of \mathbf{z} is a direct inverse inverse inverse when in a direct inverse when in \mathbf{z} An popo_{si} on n n n s is \downarrow n s po ('Juno or Minerva or Venus', 1) ('Juno or Minerva', 0.84) ('Juno or Venus', 1) ('Minerva or Venus', 1) ('Juno', 0.6) ('Minerva', 0.6) ('Venus', 1) ('', 0) \blacktriangleleft \blacktriangleleft

 $\pmb{\gamma}$

50 9 FURTHER DEVELOPMENTS

('Diana or Juno or Minerva or Venus', 1) ('Diana or Juno or Minerva', 1) ('Diana or Juno or Venus', 0.8741) ('Diana or Minerva or Venus', 0.8741) ('Juno or Minerva or Venus', 0.8659) ('Diana or Juno', 0.7796) ('Diana or Minerva', 0.7796) ('Diana or Venus', 0.6537) ('Juno or Minerva', 0.8659) ('Juno or Venus', 0.6455) ('Minerva or Venus', 0.6455) ('Diana', 0.4647) ('Juno', 0.5510) ('Minerva', 0.5510) ('Venus', 0.3306) ('', 0)

p op p o \mathbf{A} o \mathbf{B} and \mathbf{s} and \mathbf{s} b \mathbf{s}					

 Hom_xA

 $\sum_{s=s}^{\infty}$ s on \leq p in on \leq p or in \leq n in \leq \leq \leq \leq p_{res} on κ type which is determined and compiler. \mathbf{p} bases basic types employed in the present program are book \mathbf{p} are book \mathbf{p} are book \mathbf{p} and real real reals. $\frac{1}{\sqrt{2}}$ objects on a given type are obtained by $\frac{1}{\sqrt{2}}$ of a given type of a given type $\frac{1}{\sqrt{2}}$ or $\frac{1}{\sqrt{2}}$ \mathcal{A}_5 p_is n-p₁₅^o of ₁₅^o i n-p_{15^xpo} p_{is}o n-on₅ \overrightarrow{n} of \overrightarrow{q} of \overrightarrow{n} types \overrightarrow{p} types \overrightarrow{q} .

Declarations

 Apo in $\text{on}_{\text{res,ss,ss}}$ of a sequence of on_{res} are on_{res} are on_{res} n o v al . Thus v val $x = 2$; n_{B} int is not an object of type: int in on B or B $\ln 0$ introduced by the model by the word function $\ln \frac{1}{2}$ introduced by the model i fun successor $x = x + 1$; $n \t n$ n successor of the one of the int n on_{rs} \bigotimes and p_{rs}o **n** p_{rs}o **n** \bigotimes ₁₅ is arguments. Thus as arguments. Thus as arguments. Thus a fun mult $(x,y) = x * y$; int; n_{S} n on o p_{S} (int * int) -> int ¹⁴ n on_s also $n \times n$ form as in fun add $x y = x + y$; int; n_{S} a function of the p_{S} int -> (int -> int) . When n_{S} is n_{S} on, α add n_{α} is the α integer and α integer and it is given a single integer and it is given a single integer and it is given by α integer and α integer and it is given by α integer and it is given n_{res} a function from integers, the declaration from integers. Thus the d val successor = add 1; is an equivalent way of the successor function \mathbb{R}^n function \mathbb{R}^n function. Function \mathbb{R}^n p_{\bullet} and p_{\bullet} in the form val successor = fn $x \Rightarrow x + 1$; $n \rightarrow s$ of $n \rightarrow s$ of $n \rightarrow s$ of $n \rightarrow n$ be written can be w val add = fn $x \Rightarrow$ fn $y \Rightarrow x + y$; int;

The Language

Lists

 $A_{\mathcal{A},s}$ n $\frac{1}{2}$ is n order sequence of a given type. Thus $A_{\mathcal{A},s}$ is $\frac{1}{2}$ and $\frac{1}{2}$ is $\frac{1}{2$ is not only int list $\epsilon_{\rm s,s}$ on_s on p_s noted by instruction by inserting a new element and \mathbf{a} new element at the \mathbf{a} γ operation is used to denote the top denote the top denote the term is γ $[, ,] = , , [,]$ $=$ \cdots (\cdots []) $=$ \qquad $($ \qquad $($ \qquad $($ \qquad $($ \qquad n \qquad n \qquad $($ \qquad $($ \qquad \qquad n \qquad \qquad $($ \qquad \qquad \qquad \qquad $($ \qquad \qquad n \mathcal{A}_{5} , where \mathcal{B}_{5} is the pattern nil or the pattern a::l a notes the first element of \mathcal{A}_{rS} in on_{rS} on \mathcal{A}_{rS} is may be defined by definitions on \mathcal{A}_{rS} is a set of \mathcal{A}_{rS} is a se $\mathbf{r}_\mathbf{S}$ and $\mathbf{r}_\mathbf{S}$ in the sum of $\mathbf{r}_\mathbf{S}$ integers is defined recursively integers in the sum of $\mathbf{r}_\mathbf{S}$ is defined recursively integers in the sum of $\mathbf{r}_\mathbf{S}$ is defined recursively in t fun sum $nil = 0$ $|\; \text{sum} \; (a \; , \; 1) \; = \; a \; + \; \text{sum} \; 1;$ in n on o p. int list -> int . \sim $\frac{1}{5}$. The more general form of the more general form o p $n o$ on $n e^{t}$ n on iter n fun iter f u nil = u | iter f u $(a, 1) = f a$ (iter f u 1); p_a n f add n u 0 \Box on val sum = iter add 0; defines the same sum function on integer lists. (The function iter is more o n \bullet foldr or educe, $\overline{\text{I}}$ $\overline{\text{I}}$ $\overline{\text{I}}$ and \overline n filter \overline{O} n on n i n \overline{O} a₁, ..., a_n o o l o p_{φ} 'a n f _{is} on o n on f o p_{φ} 'a -> 'b , then a omap f l is the sponding \bullet is f_3a_1,\ldots,f_4a_p of \bullet is \bullet p_{\bullet} 'b . (T_{\bullet} o_{ds} 'a n^o 'b are used as type variables.) is a function of p_{\bullet} ('a -> 'b) -> ('a list -> 'b list) . Again if p no_{ts} pop ool so p. 'a niiter p is subst l of \mathbf{a} all \mathbf{a} in \mathbf{p} as in \mathbf{p} and \mathbf{p} or \mathbf{p} in \mathbf{p} in tilter is n ono p_{\bullet} ('a -> bool) -> ('a list -> 'a list) σ po_{rs} on g o f σ on on_{is t}o σ _{rs} hn op o o $_{\star}$ p. ('b -> 'c) * ('a -> 'b) -> 'a -> 'c $\begin{array}{lllllllll} \text{no} & \text{S} & \text{A} & \text{o} & \text{A} & \text{o} & \text{o} & \text{o} & \text{o} & \text{A} & \text{o} & \text{o} & \text{S} & \text{A} & \text{on} \end{array}$ $\mathbf{R}^{\mathbf{n}}$ and \mathbf{A} extra spaces, the independent are ignored. Layout $\mathbf{R}^{\mathbf{n}}$ is \mathbf{n} in \mathbf{n} is a space independent of $\mathbf{0}$ \mathbf{v} is the a matter of \mathbf{v} is \mathbf{v} or convenience.

The Code

```
Title
 \astMoebius
                                                     \ast\astLastEdit, 1 June 1
                                                     \astAuthor,
                 Peter M Williams
 \ast\astUniversity of Sussex
                                                     \astdatatype SENSE = Inf | Sup;
type LATTICE = bool list list list;
type DATUM =
    (bool list * (bool list list * bool list list)) * real;
exception hd;
fun hd nil = raise hd| hd (a, 1) = a:
fun cons a 1 = a \cdot 1;
fun iter f u nil = u| iter f u (a, 1) = f a (iter f u 1);
fun append l m = iter cons m 1;
val flat = iter append nil;
fun map f = iter (cons o f) nil;
fun filter p =iter (fn a => fn 1 => if p a then a \cup 1 else 1) nil;
val sum'r = iter (fn x => fn y => x + y) 0.0;
val inf'r =
   iter (fn x \Rightarrow fn y \Rightarrow if x < y then x else y) (1.0/0.0);
```
The Code

 \mathbf{r}

infix C ;

```
| mean 1 = \text{sum'} r \frac{1}{\text{length'}} r \frac{1}{r};
fun center nil = nil
  | center 1 =let val m = \text{mean}(\text{map } (\text{fn}(a, x) \implies x) 1)in map (fn(a,x) \implies (a,x - m)) 1 end;
fun lookup (a:bool list) nil = 0.0
  | lookup a ((b,x),i) = if a = b then x else lookup a l;
fun combine f (a, 1) (b, m) = f a b, combine f 1 m
  | combine f = - = nil;
val zero = (\text{map o map}) (\text{fn a} \Rightarrow (\text{a}, 0.0));
val add =
    (combine o combine) (fn(a,x) \Rightarrow fn(\_,y) \Rightarrow (a,x+y, real));fun mult k = (map \space o \space map) (fn(a,x) \Rightarrow (a,k*x \space real));fun profile sense lattice =
let fun insert (datum as ((b, (pos, neg)), s)) =let val x = sgn(s) * (ln(1.0 - abs s))val w = if sense = Sup then x else x
         val (S,T) =if sense = Sup then (neg,pos) else (pos,neg)
         val unit = (hd o hd o rev) lattice
         val c = union unit S
         val l = map (filter (fn a => (c C a))) lattice
         val m =iter (fn t => map (filter (fn a => not(t C a)))) 1 T
         val n =(map o map) (fn a => if b C a then (a,w) else (a,0.0)) m
         val q = (flat \ o \ map \ center) n
         fun f(a) = let val ac = a U c in (a, lookup ac q) end
    in (map o map) f lattice end
in
iter (add o insert) (zero lattice)
end;
```

```
The Code
```

```
abstype MEASURE = Measure of SENSE *((bool list * real) list list * (bool list * real) list)
with
local
fun construct sense (lattice: LATTICE) (data: DATUM list) =
let val profile = profile sense lattice data
    val measure = regularise sense profile
in Measure(sense,(profile,measure)) end
in
val infcon = construct Inf
val supcon = construct Sup
exception sense
infix ++
fun (Measure(s1,(q1,p1))) ++ (Measure(s,(q,p))) =
if s1 \Leftrightarrow s then raise sense else
let val s = s1val q = add q1 qin Measure(s,(q, regularise s q)) end
infix **
fun (Measure(s,(q,p))) ** k =
let val kq = mult k qin Measure(s,(kq, regularise s kq)) end
fun find(Measure(s,(q,p))) = p
end
end;
(***********************************************************
The exported functions have types:
```
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